Centre Number

1	2	5

CRANBROOK SCHOOL

Student Number

2020

HSC Examination

Assessment Task 3

Extension 2 Mathematics

Trial Examination

General Instructions

- Write using blue or black pen
- Diagrams drawn using pencil
- A Board-approved calculator may be used

- **Section 1**: Use the **Multiple Choice** Answer sheet for questions 1 to 10.

- **Section 2:** Please write each question in a new booklet.

- All relevant working should be shown for each question.

This paper must not be removed from the examination room

Disclaimer
The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.

Section I – Multiple Choice

10 Marks

1. Let $z = 1 + \sqrt{3}i$. What is z in exponential form?

(A) $e^{\frac{i\pi}{3}}$

(B) $e^{\frac{i\pi}{6}}$

(C) $2e^{\frac{i\pi}{3}}$

(D) $2e^{\frac{i\pi}{6}}$

2. What is the distance of the point $(2, 3, 7)$ from the x - z plane?

(A) 2 units

(B) 3 units

(C) 7 units

(D) 9 units

3. Which of the following is equivalent to $\sqrt{i^3}$?

(A) $e^{\frac{i\pi}{4}}$

(B) $e^{\frac{i\pi}{2}}$

(C) $e^{\frac{3i\pi}{4}}$

(D) $e^{-\frac{i\pi}{4}}$

4. Which of the following expressions is equal to $\int \frac{1}{x\left(\log_e x\right)^2}\,dx$?

(A) $\frac{1}{\log_e x}+C$

(B) $\frac{1}{\left(\log_e x\right)^3}+C$

(C) $\log_e\left(\frac{1}{x}\right)+C$

(D) $-\frac{1}{\log_e x}+C$

5. Let $\alpha = 1 - i$.

Which of the following is true about the value of α^{10} ?

(A) It is purely real

(B) It is purely imaginary

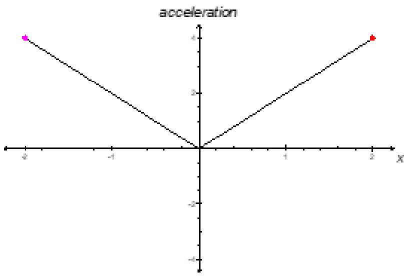
(C) 0

(D) $32\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$

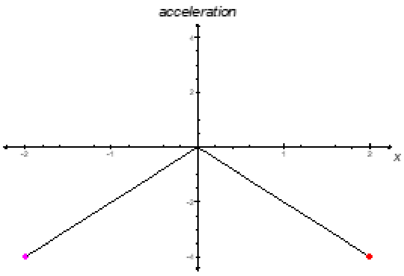
6. A particle is moving along a straight line. The displacement of the particle from a fixed point O is given by x. The graphs below show the acceleration of the particle against its displacement.

Which of the following graphs best represents the particle moving in Simple Harmonic Motion?

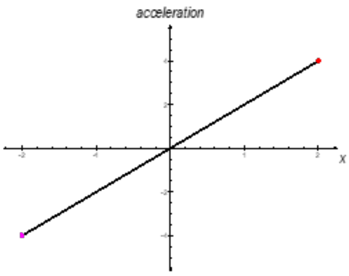
(A)



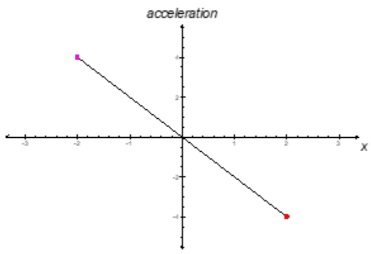
(B)



(C)



(D)



7. A particle of unit mass travels horizontally through a medium. When time $t = 0$, the particle is at point O with initial speed U. The resistance on the particle due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant.

Which expression gives the correct velocity of the particle?

(A) $\frac{1}{v} = kt + \frac{1}{U}$

(B) $v = kt + \frac{1}{U}$

(C) $\frac{1}{v} = kt$

(D) $v = kt$

8. Which expression is equal to $\int \frac{dx}{\sqrt{8 - 2x - x^2}}$?

(A) $\sin^{-1}\left(\frac{1-x}{2\sqrt{2}}\right) + C$

(B) $\sin^{-1}\left(\frac{1-x}{3}\right) + C$

(C) $\sin^{-1}\left(\frac{1+x}{2\sqrt{2}}\right) + C$

(D) $\sin^{-1}\left(\frac{1+x}{3}\right) + C$

9. A particle is moving in simple harmonic motion about the origin according to the equation $x = 3\cos nt$, where x metres is its displacement after t seconds.

Given that the particle passes through the origin with a speed of $\sqrt{3}$ ms^{-1}, what is the period of motion?

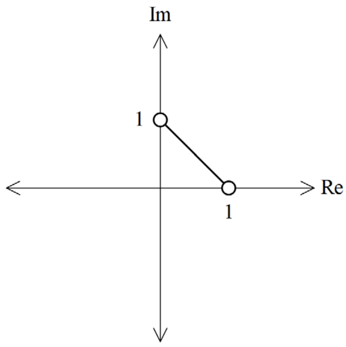
(A) $\dfrac{2\sqrt{3}\pi}{3}$ seconds

(B) $\dfrac{2\sqrt{3}}{3\pi}$ seconds

(C) $\dfrac{6\pi}{\sqrt{3}}$ seconds

(D) $\dfrac{6}{\sqrt{3}\pi}$ seconds

10. The locus of z is displayed on the Argand diagram below



Which of the following is the equation of the locus of z?

(A) $\arg\left(\dfrac{z-i}{z-1}\right)=0$

(B) $\arg\left(\dfrac{z-i}{z-1}\right)=\pm\pi$

(C) $\arg\left(\dfrac{z+i}{z+1}\right)=0$

(D) $\arg\left(\dfrac{z+i}{z+1}\right)=\pm\pi$

END OF MULTIPLE CHOICE

Section II – Extended response

90 Marks

Question 11 – Please start a new booklet. **15 Marks**

(a) Let $z = 1 + i$ and $w = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$.

(i) Express $\frac{w}{z}$ in polar form. Show all working. **2**

(ii) Hence or otherwise, express $\left(w\overline{z}\right)^8$ in the form $a + ib$ where $a, b \in \mathbb{R}$ **2**

(b) Let $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

What is the angle between the vectors, to the nearest degree? **2**

(c) Let $P(z) = z^4 - 4z^3 - 3z^2 + 50z - 52$.

Solve $P(z) = 0$ if $z = 3 - 2i$ is a root of the polynomial. **3**

(d) A line passes through the points $A(1,\ 3,\ -2)$ and $B(2,\ -1,\ 5)$.

(i) Show that the vector question of the line *AB* is given by: **2**

$$\underset{\sim}{r} = \left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \lambda_1\left(\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}\right) \quad ,\lambda_1 \in \mathbb{R}$$

(ii) Determine if the point $C(3,\ 4,\ 9)$ lies on the line. **1**

(iii) Consider a line with parametric equations $x = 1 - \lambda_2$, $y = 2 + 3\lambda_2$ and $z = -1 + \lambda_2$.

Assuming the line is neither parallel or perpendicular to *AB*, determine whether the lines intersect or are skew. **3**

END OF QUESTION 11

Question 12 - Please start a new booklet. **15 Marks**

(a) Consider the equation $z^2 - 2(1+2i)z + (1+i) = 0$.

(i) Show that $\left(z-(1+2i)\right)^2 = -4+3i$ **1**

(ii) Hence solve $z^2 - 2(1+2i)z + (1+i) = 0$ **3**

(b) Sketch the intersection of the regions defined by: **2**

$$|z-2i| \le 1 \text{ and } 0 < Arg(z-2i) \le \frac{3\pi}{4}.$$

(c) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{3+5\cos x}\,dx$.

(i) Using the substitution $t = \tan\dfrac{x}{2}$ show that $I = \displaystyle\int_0^1 \frac{2}{4-t^2}\,dt$. **2**

(ii) Hence find the value of *I*. Give your answer in the form $\ln\sqrt{k}$ where *k* is a positive integer. **2**

(d) On the Argand diagram below, points A and B correspond to the complex numbers $z_1 = 5 + 3i$ and $z_2 = 3 - i$. M is the midpoint of the interval AB and QM is drawn such that it is perpendicular to AB and $QM = AM = BM$. Let Q represent the complex number ω.

Find all possible values of ω in the form $a + ib$. **3**

(e) Evaluate $\displaystyle\int_0^3 u\sqrt{u+1}\,du$ **2**

END OF QUESTION 12

Question 13 – Please start a new booklet. **15 Marks**

(a) Consider the following.

(i) Express $\dfrac{-x^2+2x+5}{\left(x^2+2\right)\left(1-x\right)}$ in the form $\dfrac{ax+b}{x^2+2}+\dfrac{c}{1-x}$ **2**

(ii) Hence find $\displaystyle\int \frac{-x^2+2x+5}{\left(x^2+2\right)\left(1-x\right)}\,dx$. **2**

(b) A particle of mass M kilograms is projected vertically upward with a velocity of $120\,\text{ms}^{-1}$. The air resistance acting on the particle is $3Mv$ newtons where v is the velocity of the particle.

(i) Show that if the acceleration due to gravity is 10 ms^{-2}, the equation of motion is given by $\ddot{x}=-\left(10+3v\right)$. **1**

(ii) Find the maximum height reached by the particle, correct to the nearest metre. **3**

(iii) Find the time at which the particle reaches its maximum height, correct to one decimal place. **2**

(c) A sphere S_1 with centre $\underset{\sim}{c} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ passes through $\underset{\sim}{a} = 4\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$.

(i) Find the Cartesian equation of S_1. **2**

(ii) A second sphere, S_2, has equation $(x-2)^2 + (y-2)^2 + (z-5)^2 = 1$. Find the equation of the circle in which S_1 and S_2 intersect and state the centre and radius of this circle. **3**

END OF QUESTION 13

Question 14 – Please start a new booklet. **15 Marks**

(a) A particle is moving such that its speed (in ms^{-1}) is given by $v^2 = 2 - x - x^2$, where x is the displacement of the particle from a fixed-point O.

(i) Show that the particle in moving in Simple Harmonic Motion. **2**

(ii) What is the maximum distance of the particle from O? **1**

(b) The price, p, of fuel rises and falls in Simple Harmonic Motion according to the equation $p = \dfrac{3 + \sin\left(\dfrac{\pi t}{7}\right) + \cos\left(\dfrac{\pi t}{7}\right)}{2}$, where the price is measured in dollars and t is the numbers of days after 9 am on Sunday.

(i) What is the amplitude and period of the fuel price? **3**

(ii) What is the price of fuel, to the nearest cent, at 9 am on Monday? **1**

(iii) At what time and day, correct to the nearest hour, will the fuel price first be at a minimum?

3

(c) A particle is moving such that $\ddot{x} = 2x^3 + 6x^2 + 4x$. Initially $x = 1$ and $v = -3$.

(i) Show that $v = -x(x+2)$. **2**

(ii) Find an expression for x in terms of t. **2**

(iii) Hence, find the limiting position of the particle. **1**

END OF QUESTION 14

Question 15 – Please start a new booklet. **15 Marks**

(a) Find $\int e^{-x}\sin(-x)\,dx$. **3**

(b) A projectile is fired from ground level with an initial velocity of u ms^{-1} at an angle of θ to the horizontal. The air resistance is directly proportional to the velocity, with k the constant of proportionality.

Assume that the equations of motion are:

$$x = \frac{u\cos\theta}{k}\left(1-e^{-kt}\right)$$

$$y = \frac{10+ku\sin\theta}{k^2}\left(1-e^{-kt}\right)-\frac{10t}{k}$$

where $(x,\ y)$ are the coordinates of the projectile at time t seconds. Do NOT prove these equations.

A projectile is fired at an angle of 60°, with initial velocity $10\sqrt{3}$ ms^{-1} and $k = 0.4$.

(i) Find the time when the projectile reaches its greatest height. **2**

(ii) The projectile hits the ground when $t \approx 2.6$ seconds. Find the magnitude and direction of the velocity of the projectile when it hits the ground. **3**

(c) Consider the equation $z^5 = 1$.

(i) Write down, in polar form, the five roots of $z^5 = 1$. **2**

(ii) Show that for $z \neq 1$: **2**

$$\frac{z^5 - 1}{z - 1} = \left(z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right)$$

(iii) Deduce that $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ are roots of the equation $4x^2 + 2x - 1 = 0$.

3

END OF QUESTION 15

Question 16 – Please start a new booklet. **15 Marks**

(a) Let the line l have parametric equations $x = 3 - \lambda$, $y = 2 + 2\lambda$ and $z = 5 - 2\lambda$.

Find the distance from the line to the point $\underset{\sim}{R} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ **3**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \ dx$.

(i) Show that $I_n = \left(\dfrac{n-1}{n+2}\right) I_{n-2}$ for $n \geq 2$. **4**

(ii) Hence show that $I_2 = \dfrac{\pi}{16}$. **1**

(c) Consider the equation $z^5 = (z+1)^5$ where $z \in \mathbb{C}$.

(i) Explain why this equation does NOT have five roots? **1**

(ii) Solve $z^5 = (z+1)^5$, giving your answer in the form $a + bi\cot\theta$ where $a, b, \theta \in \mathbb{R}$ **4**

(iii) Describe the geometrical relationship between the roots of the equation $z^5 = (z+1)^5$ and the roots of the equations $iz^5 = (iz+1)^5$. Provide some mathematical working to justify your answer. **2**

END OF EXAM

Multi Choice

1. C
2. B
3. C
4. D
5. B
6. D
7. A
8. D
9. C
10. B

Q11.) a.) $z = 1+i$ $\quad w = \text{cis}\,\frac{\pi}{6}$

$z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ (1) (for converting z or w correctly)

i.) $\frac{w}{z} = \frac{\text{cis}\,\frac{\pi}{6}}{\sqrt{2}\,\text{cis}\,\frac{\pi}{4}}$

$\frac{w}{z} = \frac{\sqrt{2}}{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$ (1)

ii.) $\frac{w}{z} = \frac{w\bar{z}}{|z|^2}$

$\therefore \frac{w\bar{z}}{|z|^2} = \frac{\sqrt{2}}{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$

$w\bar{z} = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$

$(w\bar{z})^8 = 2^4\,\text{cis}\left(\frac{-8\pi}{12}\right)$ (1) (by DMT)

$= 16\,\text{cis}\left(\frac{-2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -8-8\sqrt{3}\,i$ (1)

Bad angle ⇒ Can't convert 0

b.) $\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$

$|\underset{\sim}{a}| = \sqrt{14} \qquad |\underset{\sim}{b}| = \sqrt{6}$ ① for either

$\cos\theta = \frac{5}{\sqrt{14}\sqrt{6}}$

$\theta = 57°$ (nearest degree) ①

c.) If $z = 3-2i$ is a root of $P(z)$

then $z = 3+2i$ is a root by conjugate root theorem ①

$(z-(3-2i))(z-(3+2i))$

$= z^2 - (3+2i)z - (3-2i)z + 9 + 4$

$= z^2 - 6z + 13$

$(z^2-6z+13)(z^2+bz-4) \equiv z^4 - 4z^3 - 3z^2 + 50z - 52$

Equating coeff of z^3

$b - 6 = -4$

$\therefore b = 2$ ①

$\therefore P(z) = (z^2-6z+13)(z^2+2z-4)$

When $P(z) = 0$

$\boxed{z = 3 \pm 2i} \qquad z = \frac{-2 \pm \sqrt{20}}{2}$

$\boxed{z = -1 \pm \sqrt{5}}$ ①

d.) i.) $\vec{AB} = -\vec{OA} + \vec{OB}$

$= -(1,3,-2) + (2,-1,5)$

$= (1,-4,7)$ ①

$\therefore$ Eq of line

$\underset{\sim}{r} = \vec{OA} + \lambda_1 \vec{AB}$ ① must have this line

$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k} + \lambda_1(\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k})$

ii.) When $\lambda_1 = 2$

$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k} + 2(\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k})$

$= 3\underset{\sim}{i} - 5\underset{\sim}{j} + 12\underset{\sim}{k}$ ① with working

No $(3,4,9)$ does not lie on the line

iii.) If intersect then there exists a solution to

$1 + \lambda_1 = 1 - \lambda_2 \Rightarrow \lambda_1 + \lambda_2 = 0$ — ①

$3 - 4\lambda_1 = 2 + 3\lambda_2 \Rightarrow 4\lambda_1 + 3\lambda_2 = 1$ — ②

$-2 + 7\lambda_1 = -1 + \lambda_2 \Rightarrow 7\lambda_1 - \lambda_2 = 1$ — ③ ①

① + ③ $\quad 8\lambda_1 = 1$

$\lambda_1 = \frac{1}{8} \qquad \therefore \lambda_2 = -\frac{1}{8}$

Sub $\lambda_1 = \frac{1}{8}$, $\lambda_2 = -\frac{1}{8}$ into ②

$LHS = 4(\frac{1}{8}) + 3(-\frac{1}{8})$

$= \frac{1}{8} \neq 1 = RHS$ ①

$\therefore$ Lines are skew. ①

Question 12

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(a) (i) $z^2 - 2(1+2i)z + (1+i) = 0$

$z^2 - 2(1+2i)z = -(1+i)$

$+(1+2i)^2 \qquad + (1+2i)^2$

$z^2 - 2(1+2i)z + (1+2i)^2 = -1 - i + (1+2i)^2$

$[z - (1+2i)]^2 = -1 - i + (1 + 4i - 4)$

Must show these steps.

$= -4 + 3i$ As required

(ii) $z^2 - 2(1+2i)z + (1+i) = 0$

$\therefore$ solve $[z - (1+2i)]^2 = -4 + 3i$

Need $\sqrt{-4+3i}$

let $x + iy = \pm\sqrt{-4+3i}$

$\therefore x^2 - y^2 + (2xy)i = -4 + 3i$

$\therefore x^2 - y^2 = -4 \qquad 2xy = 3$ (By comparing real & imaginary)

$y = \dfrac{3}{2x}$

(1) establishing these relationships

$\therefore x^2 - \dfrac{9}{4x^2} = -4$

$4x^4 - 9 = -16x^2$

$4x^4 + 16x^2 - 9 = 0$

$(2x^2 + 9)(2x^2 - 1) = 0$

$\therefore x^2 = -\dfrac{9}{2}$ or $\dfrac{1}{2}$

As $x \in \mathbb{R}$, $x = \pm\sqrt{\dfrac{1}{2}}$

$\therefore y = \pm \dfrac{3\sqrt{2}}{2} \qquad \therefore x + iy = \pm\left(\dfrac{\sqrt{2} + 3\sqrt{2}}{2}\right)$ (1)

$\therefore z - (1+2i) = \pm \dfrac{\sqrt{2} + 3\sqrt{2}}{2}$

$\therefore z = 1 + 2i \pm \left(\dfrac{\sqrt{2} + 3\sqrt{2}}{2}\right)$

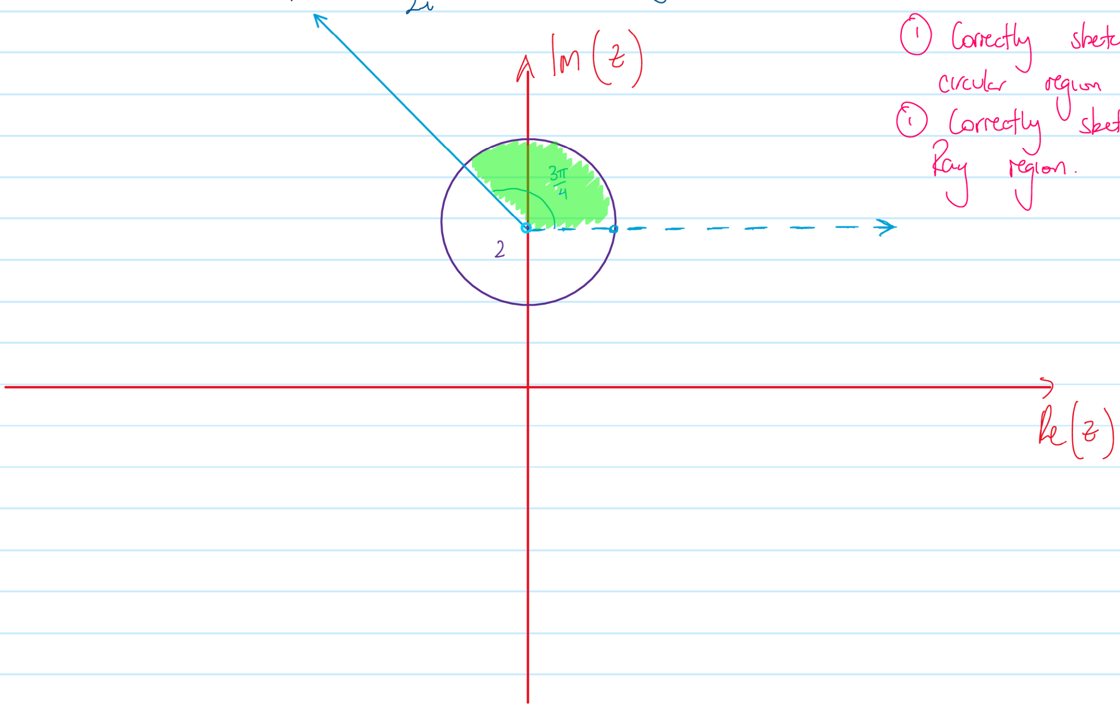
$\therefore 2+\sqrt{2} + (4+3\sqrt{2})i \quad , \quad (2-\sqrt{2}) + (4-3\sqrt{2})i$ (1)

$$\therefore \quad z = 1+2i \pm \left(\frac{\sqrt{2}+3\sqrt{2}}{2}\right)$$

$$z = \frac{2+\sqrt{2}+(4+3\sqrt{2})i}{2} \quad \& \quad \frac{(2-\sqrt{2})+(4-3\sqrt{2})i}{2}$$ (1)

(b) $|z-2i| \leq 1$ Inside of a circle, radius one centre @ $2i$

$0 < \operatorname{Arg}(z-2i) \leq \frac{3\pi}{4}$ In between 2 rays from $2i$



(1) Correctly sketch circular region
(1) Correctly sketches Ray region.

(c) (i) $$\int_0^{\frac{\pi}{2}} \frac{2}{3+5\cos x} \, dx$$

Let $t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1}(t)$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$\therefore dx = \frac{2dt}{1+t^2}$

(1)

When $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

$$\therefore I = \int_0^1 \frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2}{\frac{3+3t^2+5-5t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{\cancel{1+t^2}}$$ (1)

$$= \int_0 \frac{2}{\frac{3+3t^2+5-5t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{\cancel{1+t^2}}$$

$$= \int_0^1 \frac{4}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{2}{4-t^2}\,dt \quad \text{As required.}$$

(ii) $$\int_0^1 \frac{2}{4-t^2}\,dt$$

Let $\frac{A}{2-t} + \frac{B}{2+t} = \frac{2}{4-t^2}$

$\therefore A(2+t) + B(2-t) = 2$ ①

@ $t=2$, $4A = 2$, $A = \frac{1}{2}$

$t=-2$, $4B = 2$, $B = \frac{1}{2}$

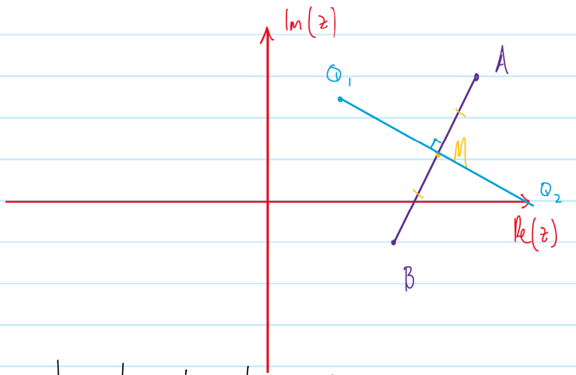
$$\therefore I = \int_0^1 \left[\frac{1}{2}\left(\frac{1}{2-t} + \frac{1}{2+t}\right)\right] dt$$

$$= \frac{1}{2}\Big[\ln(2+t) - \ln(2-t)\Big]_0^1$$

$$= \frac{1}{2}\Big[\big(\ln(3) - \ln(1)\big) - (\ln 2 - \ln 2)\Big]$$ ①

$$= \ln\sqrt{3}$$

(d)



As $|QM| = |AM|$ & QM is perpendicular to AB

then $\overrightarrow{MQ} = i\,\overrightarrow{MA}$ & $\overrightarrow{MQ} = -i\overrightarrow{MA}$

① establish a relationship between known vectors & the vector $\overrightarrow{MQ}$

$\overrightarrow{OM} = 4+i$

$\therefore \overrightarrow{OA} = \overrightarrow{OM} + \overrightarrow{MA}$

$\therefore \overrightarrow{MA} = 5+3i - (4+i)$

$= 1+2i$

Question 12

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(a) (i) $z^2 - 2(1+2i)z + (1+i) = 0$

$$z^2 - 2(1+2i)z = -(1+i)$$

$$+(1+2i)^2 \qquad +(1+2i)^2$$

$$z^2 - 2(1+2i)z + (1+2i)^2 = -1-i+(1+2i)^2$$

$$[z-(1+2i)]^2 = -1-i+(1+4i-4)$$

$$= -4+3i \qquad \text{As required}$$

Must show these steps.

(ii) $z^2 - 2(1+2i)z + (1+i) = 0$

$\therefore$ solve $[z-(1+2i)]^2 = -4+3i$

Need $\sqrt{-4+3i}$

let $x+iy = \pm\sqrt{-4+3i}$

$\therefore x^2 - y^2 + (2xy)i = -4+3i$

$\therefore x^2 - y^2 = -4 \qquad 2xy = 3 \qquad$ (By comparing real & imaginary)

$y = \dfrac{3}{2x}$

(1) establishing these relationships

$$\therefore x^2 - \frac{9}{4x^2} = -4$$

$$4x^4 - 9 = -16x^2$$

$$4x^4 + 16x^2 - 9 = 0$$

$$(2x^2+9)(2x^2-1) = 0$$

$$\therefore x^2 = -\frac{9}{2} \text{ or } \frac{1}{2}$$

As $x \in \mathbb{R}$, $x = \pm\sqrt{\frac{1}{2}}$

$\therefore y = \pm\dfrac{3\sqrt{2}}{2} \qquad \therefore x+iy = \pm\left(\dfrac{\sqrt{2}+3\sqrt{2}}{2}\right)$ (1)

$$\therefore z-(1+2i) = \pm\frac{\sqrt{2}+3\sqrt{2}}{2}$$

$$\therefore z = 1+2i \pm \left(\frac{\sqrt{2}+3\sqrt{2}}{2}\right)$$

Q13.) a.) i.) $\frac{-x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{ax+b}{x^2+2} + \frac{c}{1-x}$

$$-x^2+2x+5 \equiv (ax+b)(1-x) + c(x^2+2)$$

1

When $x = 1$

$$-(1)^2+2(1)+5 = 3c$$

$$\therefore c = 2$$

Equating co-eff of x^2

$$-1 = -a + c$$

$$a = 1 + 2$$

$$a = 3$$

Equating constant

$$5 = b + 2c$$

$$5 = b + 2(2)$$

$$\therefore b = 1$$

1 for a, b, c

$$\therefore \frac{-x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{3x+1}{x^2+2} + \frac{2}{1-x}$$

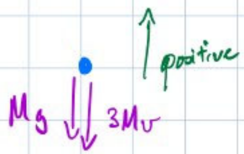
ii.) $\int \frac{-x^2+2x+5}{(x^2+2)(1-x)}\,dx = \int \frac{3x+1}{x^2+2} + \frac{2}{1-x}\,dx$

$$= \frac{3}{2}\int \frac{2x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx + 2\int \frac{1}{1-x}\,dx$$

$$= \frac{3}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) - 2\ln|1-x| + c$$

1 error: 1/2

b.) i.)



$$m\underset{\sim}{a} = \underset{\sim}{F}$$

$$M\ddot{x} = -M(10) - 3Mv$$ 1

$$\therefore \ddot{x} = -(10+3v)$$

ii.) $v\frac{dv}{dx} = -(10+3v)$

$$\int \frac{v}{10+3v}\,dv = \int -1\,dx$$

$$\frac{1}{3}\int \frac{3v+10-10}{10+3v}\,dv = -x + c$$

$$\frac{1}{3}\int 1 - \frac{10}{10+3v}\,dv = -x + c$$

$$\frac{1}{3}\left(v - \frac{10}{3}\ln|10+3v|\right) = -x + c$$ 1

$$3v - 10\ln|10+3v| = -9x + c_2$$

When $x=0$, $v=120$

$$360 - 10\ln|370| = c_2$$ (1)

$$3v - 10\ln|10+3v| = -9x + 360 - 10\ln(370)$$

$$9x = 360 - 3v + 10\ln\left|\frac{10+3v}{370}\right|$$

When $v=0$

$$9x = 360 + 10\ln\left|\frac{10}{370}\right|$$

$$x = 40 + \frac{10}{9}\ln\left|\frac{10}{370}\right| \simeq 36$$

$\therefore$ Max height 36m (1) metres

iii.) $\frac{dv}{dt} = -(10+3v)$

$$\int \frac{1}{10+3v}\,dv = \int 1\,dt$$

$$\frac{1}{3}\ln|10+3v| = -t + c$$ (1)

When $t=0$, $v=120$

$$\frac{1}{3}\ln|370| = c$$

$$t = \frac{1}{3}\ln|370| - \frac{1}{3}\ln|10+3v|$$

$$t = \frac{1}{3}\ln\left|\frac{370}{10+3v}\right|$$

Max height when $v=0$

$$t = \frac{1}{3}\ln|37|$$

(1)

$\therefore$ Max height at $t=1.2$ seconds.

c) i.) Let $\underset{\sim}{x} = (x, y, z)$

$$|\underset{\sim}{x} - \underset{\sim}{c}| = |\underset{\sim}{a} - \underset{\sim}{c}|$$

$$|\underset{\sim}{a} - \underset{\sim}{c}| = \sqrt{(4-2)^2 + (4-2)^2 + (4-2)^2}$$

$$|\underset{\sim}{a} - \underset{\sim}{c}| = \sqrt{12} \quad (1)$$

$\therefore S_1: (x-2)^2 + (y-2)^2 + (z-2)^2 = 12 \quad (1) \quad -(1)$

ii.) $(x-2)^2 + (y-2)^2 + (z-5)^2 = 1 \quad -(2)$

$(1)-(2)$: $(z-2)^2 - (z-5)^2 = 11$

$$z^2 - 4z + 4 - (z^2 - 10z + 25) = 11$$

$$6z - 21 = 11$$

$$6z = 32$$

$$z = \frac{32}{6} = \frac{16}{3} \quad (1)$$

$\therefore$ Intersect at $z = \frac{16}{3}$

When $z = \frac{16}{3}$

$$(x-2)^2 + (y-2)^2 + \left(\frac{16}{3} - 2\right)^2 = 12$$

$$(x-2)^2 + (y-2)^2 = \frac{8}{9}$$

$\therefore$ Equation of circle

$$(x-2)^2 + (y-2)^2 = \frac{8}{9}, \quad z = \frac{16}{3}$$

Center: $\left(2, 2, \frac{16}{3}\right)$ radius: $\frac{2\sqrt{2}}{3}$

Eq + center + radius: $\frac{1}{2}$ (1)

(incl z)

Question 14

Friday, 21 August 2020 8:35 am

(a) (i) $v^2 = 2 - x - x^2$

$= 2 + \frac{1}{4} - \frac{1}{4} - x - x^2$

$v^2 = \frac{9}{4} - \left(x + \frac{1}{2}\right)^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\therefore \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\left[\frac{9}{4} - \left(x + \frac{1}{2}\right)^2\right]\right)$

$= \frac{1}{\cancel{2}} \times -\left(x + \frac{1}{2}\right)^1 \times \cancel{2} \times 1$

$= -\left(x + \frac{1}{2}\right)$ 1

As $\ddot{x}$ is proportional to, but in the opposite direction of, the displacement then the particle is moving in SHM. 1 or equivalent.

(ii) Max displacement occurs when $v = 0$

$0 = \frac{9}{4} - \left(x + \frac{1}{2}\right)^2$

$x + \frac{1}{2} = \pm\frac{3}{2}$

$x = \frac{1}{2}$ or -2

But, distance is scalar. $\therefore$ Maximum distance is 2 units. 1

(b) $p = \dfrac{3 + \sin\left(\frac{\pi t}{7}\right) + \cos\left(\frac{\pi t}{7}\right)}{2}$

(i) let $A = \sin\left(\frac{\pi t}{7}\right) + \cos\left(\frac{\pi t}{7}\right)$

& $A = B\cos\left(\frac{\pi t}{7} - \alpha\right)$

$\therefore = \underbrace{B}\cos\left(\frac{\pi t}{7}\right)\underbrace{\cos\alpha} + \underbrace{B}\sin\left(\frac{\pi t}{7}\right)\underbrace{\sin\alpha}$

$\therefore \quad B\cos\alpha = 1 \qquad B\sin\alpha = 1$

$\therefore \quad B = \sqrt{2} \quad \& \quad \alpha = \frac{\pi}{4}$

$\therefore \quad p = \dfrac{3 + \sqrt{2}\cos\left(\frac{\pi t}{7} - \frac{\pi}{4}\right)}{2}$

1 This or equivalent

$\therefore$ Amplitude of motion is $\frac{\sqrt{2}}{2}$ 1

Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{\pi/7}$

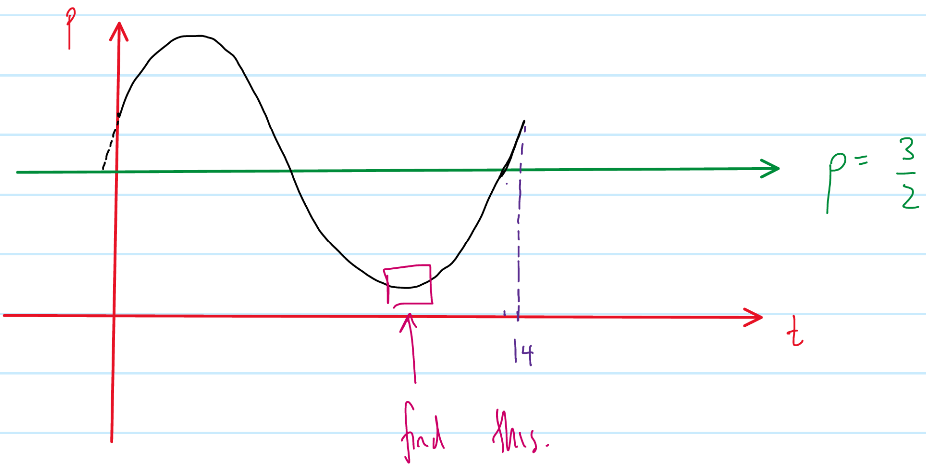
$= 14$ 1

(ii) 9 am Monday occurs when $n = 1$

$\therefore \quad p = \dfrac{3 + \sqrt{2}\left(\cos\left(\frac{\pi}{7} - \frac{\pi}{4}\right)\right)}{2}$

$= \$2.83$ (2dp) 1

(iii)



Minimum will occur when $\cos\left(\frac{\pi t}{7} - \frac{\pi}{4}\right)$ is a minimum. (1)

$$\therefore \cos\left(\frac{\pi t}{7} - \frac{\pi}{4}\right) = -1$$

$$\frac{\pi t}{7} - \frac{\pi}{4} = \pi$$

$$\frac{\pi t}{7} = \frac{5\pi}{4}$$

$$t = 35/4 \text{ days}$$

$$= 8 \text{ days, } 18 \text{ hours.} \quad (1)$$

$\therefore$ It will be a minimum on the second tuesday at 3am. (1)

(c) $\ddot{x} = 2x^3 + 6x^2 + 4x \qquad x = 1,\ t = 0,\ v = -3$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 6x^2 + 4x$$

$$\frac{1}{2}v^2 = \int 2x^3 + 6x^2 + 4x \, dx$$

$$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^3 + 2x^2 + C_1$$

v

$$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^3 + 2x^2 + C_1$$

$$v^2 = x^4 + 4x^3 + 4x^2 + c_2 \quad (1)$$

When $v = -3$, $x = 1$

$$9 = 1 + 4 + 4 + c_2$$

$$c_2 = 0$$

$$\therefore v^2 = x^4 + 4x^3 + 4x^2$$

$$= x^2\left(x^2 + 4x + 4\right)$$

$$= x^2(x+2)^2$$

$$V = -x(x+2)$$ As when $x = 1$, $v = -3$ hence negative square root. (1)

(ii) $$\frac{dx}{dt} = -x(x+2)$$

$$\frac{dt}{dx} = \frac{-1}{x(x+2)}$$

let $\frac{A}{x} + \frac{B}{x+2} = \frac{-1}{x(x+2)}$

$Ax + 2A + Bx = -1$

$\therefore A + B = 0, \quad 2A = -1$

$\therefore A = -\frac{1}{2}, \quad B = \frac{1}{2}$

$$\frac{dt}{dx} = \frac{1}{2}\left[\frac{1}{x+2} - \frac{1}{x}\right]$$

$$\int_0^t dt = \frac{1}{2}\int_1^x \frac{1}{x+2} - \frac{1}{x}\, dx$$

$$t = \frac{1}{2}\left[\ln\left[\frac{x+2}{x}\right]\right]_1^x \quad (1)$$

$$= \frac{1}{2}\left[\ln\left[\frac{x+2}{x}\right] - \ln 3\right]$$

$$2t = \ln\left(\frac{x+2}{3x}\right)$$

$$2t = \ln\left(\frac{x+2}{3x}\right)$$

$$\ln\left(\frac{3x}{x+2}\right) = -2t$$

$$\frac{3x}{x+2} = e^{-2t}$$

$$3x = e^{-2t}(x+2)$$

$$3x - xe^{-2t} = 2e^{-2t}$$

$$x(3-e^{-2t}) = 2e^{-2t}$$

$$\therefore \quad x = \frac{2e^{-2t}}{3-e^{-2t}}$$

(iii) As $t \to \infty \quad e^{-2t} \to 0$

$\therefore \quad x \to 0$ 1

Q15.) a.) $\int e^{-x} \sin(-x)\, dx$

$u = \sin(-x) \qquad v' = e^{-x}$

$u' = -\cos(-x) \qquad v = -e^{-x}$

$$\int e^{-x}\sin(-x)\,dx = -e^{-x}\sin(-x) - \int e^{-x}\cos(-x)\,dx$$

1

$u = \cos(-x) \qquad v' = e^{-x}$

$u' = \sin(-x) \qquad v = -e^{-x}$

$$\int e^{-x}\sin(-x)\,dx = -e^{-x}\sin(-x) - \left[-e^{-x}\cos(-x) - \int -e^{-x}\sin(-x)\,dx\right]$$

1

$$\int e^{-x}\sin(-x)\,dx = -e^{-x}\sin(-x) + e^{-x}\cos(-x) - \int e^{-x}\sin(-x)\,dx$$

$$2\int e^{-x}\sin(-x)\,dx = e^{-x}\cos(-x) - e^{-x}\sin(-x)$$

$$\therefore \int e^{-x}\sin(-x)\,dx = \frac{e^{-x}\cos(-x) - e^{-x}\sin(-x)}{2} + C$$

1

b.) i.) $y = \dfrac{10 + \cdot 4(10\sqrt{3})\sin 60}{\cdot 4^2}\left(1 - e^{-\cdot 4t}\right) - \dfrac{10t}{\cdot 4}$

$$y = 100\left(1 - e^{-\cdot 4t}\right) - 25t$$

$$\dot{y} = 40e^{-\cdot 4t} - 25$$

1

When $\dot{y} = 0$

$$0 = 40e^{-\cdot 4t} - 25$$

$$e^{-\cdot 4t} = \frac{25}{40}$$

$$-\cdot 4t = \ln\frac{25}{40}$$

$$t = -\frac{1}{\cdot 4}\ln\left(\frac{25}{40}\right)$$

$$\simeq 1\cdot 175 \text{ sec.}$$

1

b) $x = \frac{10\sqrt{3}\cos 60^\circ}{.4}\left(1-e^{-.4t}\right)$

$x = \frac{25\sqrt{3}}{2}\left(1-e^{-.4t}\right)$

$\dot{x} = 5\sqrt{3}e^{-.4t}$ 1

When $t = 2.6$

$\dot{y} = -10.86$ $\quad \dot{x} = 3.06$ 1 for both

∴ Magnitude: $\sqrt{127.3032} = 11.28\,ms^{-1}$

$\tan\theta = \frac{10.86}{3.06} = 74.3^\circ$ 1

At an angle of 74.3° to the ground.

c) i) Let $z = r\,\text{cis}\,\theta$

$(r\,\text{cis}\,\theta)^5 = 1$

$r^5\,\text{cis}\,5\theta = 1$ (by DMT)

$\therefore r = 1$ $\quad 5\theta = \arg(1)$

$5\theta = 0, \pm 2\pi, \pm 4\pi$

$\therefore \theta = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$

$\therefore z_1 = 1$, $z_2 = \text{cis}\,\frac{2\pi}{5}$, $z_4 = \text{cis}\,\frac{4\pi}{5}$ 2 for all

$z_3 = \text{cis}\,\frac{-2\pi}{5}$ $\quad z_5 = \text{cis}\left(-\frac{4\pi}{5}\right)$ 1 1 error

ii) As z_1, z_2, z_3, z_4, z_5 are roots to equation $z^5 - 1 = 0$

$\Rightarrow z^5 - 1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$

As $\bar{z}_2 = z_3$, $\bar{z}_4 = z_5$

$z^5 - 1 = (z-1)(z-z_2)(z-\bar{z}_2)(z-z_4)(z-\bar{z}_4)$ 1

$= (z-1)\left(z-(z_2+\bar{z}_2)z + z_2\bar{z}_2\right)\left(z-(z_4+\bar{z}_4)z + z_4\bar{z}_4\right)$

1

$= (z-1)\left(z - 2\text{Re}(z_2)z + |z_2|^2\right)\left(z - 2\text{Re}(z_4)z + |z_4|^2\right)$

$z^5 - 1 = (z-1)\left(z - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$

$\frac{z^5-1}{(z-1)} = \left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$

iii.) $LHS = z^4+z^3+z^2+z+1$

$RHS = z^4 - \left(2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)\right)z^3 + \left(2+4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)\right)z^2 - \left(2\cos\left(\frac{2\pi}{5}\right)+2\cos\frac{4\pi}{5}\right)z+1$

* ① for recognising need to equate coefficients

① for both for establishing either by Equating co-eff or sum of roots for quartic.

$-\left(2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}\right)=1 \Rightarrow \cos\frac{2\pi}{5} = -\frac{1}{2}-\cos\frac{4\pi}{5}$ — ①

$4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)=-1$ — ②

Sub ① into ②

$$\left.\begin{aligned} &4\left(-\frac{1}{2}-\cos\frac{4\pi}{5}\right)\cos\frac{4\pi}{5}=-1 \\ &-2\cos\frac{4\pi}{5}-4\cos^2\frac{4\pi}{5}=-1 \end{aligned}\right\}$$

$\therefore 0 = 4\cos^2\frac{4\pi}{5}+2\cos\frac{4\pi}{5}-1$

Let $x=\cos\frac{4\pi}{5}$

$0=4x^2+2x-1$ where $x=\cos\frac{4\pi}{5}$ is a root

Similarly $\cos\frac{4\pi}{5} = -\frac{1}{2}-\cos\frac{2\pi}{5}$ — ①'

Sub ①' into ②

$4\cos\frac{2\pi}{5}\left(-\frac{1}{2}-\cos\frac{2\pi}{5}\right)=-1$

$-2\cos\frac{2\pi}{5}-4\cos^2\frac{2\pi}{5}=-1$

$0=4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1$

Let $x=\cos\frac{2\pi}{5}$

$\therefore 0=4x^2+2x-1$ where $x=\cos\frac{2\pi}{5}$ is a root.

① for establishing $x=\cos\frac{4\pi}{5}$, $x=\cos\frac{2\pi}{5}$ as roots to $4x^2+2x-1=0$

$\therefore x=\cos\frac{2\pi}{5}$, $x=\cos\frac{4\pi}{5}$ are roots to $4x^2+2x-1=0$

*Alternatively

Let α and β be the roots to

$4x^2+2x-1=0$

$\alpha+\beta = -\frac{2}{4} \Rightarrow \alpha+\beta=-\frac{1}{2}$

$\alpha\beta = -\frac{1}{4}$

RTP: $\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$ and $\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{4}$

From ii.)

$\frac{z^5-1}{z-1} = (z-z_2)(z-\overline{z_2})(z-z_4)(z-\overline{z_4})$

$\therefore z^4+z^3+z^2+z+1 = (z-z_2)(z-\overline{z_2})(z-z_4)(z-\overline{z_4})$

Sum of roots for $z^4+z^3+z^2+z+1=0$

$\therefore z_2+\overline{z_2}+z_4+\overline{z_4}=-1$

$\therefore 2Re(z_2)+2Re(z_4)=-1$

$\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$

The equate coeff for 2nd part as above.

Question 16

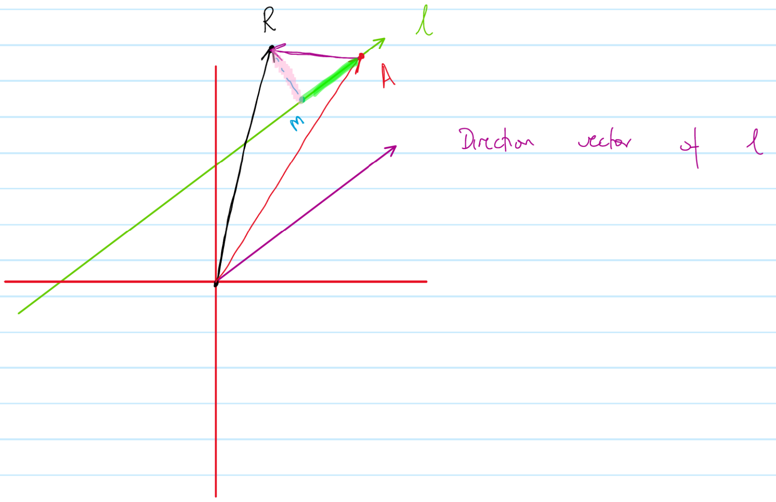
Friday, 21 August 2020 8:35 am

(a) l , $x = 3-\lambda$, $y = 2+2\lambda$, $z = 5-2\lambda$

Direction vector of l : $\underset{\sim}{L} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ Point on the line : let $\underset{\sim}{A} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

$\underset{\sim}{R} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

Find the vector projection of R onto l



let $\underset{\sim}{M}$ represent the point on l such that $\overrightarrow{RM}$ is perpendicular to l

$\therefore$ $|\overrightarrow{RM}|$ is the distance of $\underset{\sim}{R}$ from l

$\overrightarrow{AM}$ is the vector projection of $\overrightarrow{AR}$ onto l (the direction vector of l) 1

$$\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA}$$
$$= \underset{\sim}{R} - \underset{\sim}{A}$$
$$= \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$

$$\overrightarrow{AM} = \text{proj}_{\underset{\sim}{L}} \overrightarrow{AR}$$
$$= \left[\frac{\overrightarrow{AR} \cdot \underset{\sim}{L}}{|\underset{\sim}{L}|} \right] \times \frac{\underset{\sim}{L}}{|\underset{\sim}{L}|}$$

$$= \left[\frac{\begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}}{3} \right] \times \frac{\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}}{3}$$

$$= -\frac{4}{3} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \quad 1$$

$$\vec{AR} = \vec{AM} + \vec{MR}$$

$$\therefore \vec{MR} = \vec{AR} - \vec{AM}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} + \left[-\frac{4}{3} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 4/3 \\ -8/3 \\ -10/3 \end{bmatrix}$$

$$|\vec{MR}| = \sqrt{20} = 2\sqrt{5} \quad 1$$

(b) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \, dx$

Show $I_n = \left(\frac{n-1}{n+2} \right) I_{n-2} \quad n \geq 2$

$$I_n = \int_0^{\frac{\pi}{2}} \underline{\cos^{n-1} x} \cdot \underline{\cos x \sin^2 x} \, dx$$

$$u = \cos^{n-1} x \qquad v = \frac{\sin^3 x}{3}$$

$$\frac{du}{dx} = (n-1) \cos^{n-2} x \cdot -\sin x \qquad \frac{dv}{dx} = \cos x \sin^2 x \quad 1$$

$$\therefore I_n = \left[\frac{\cos^{n-1} x \cdot \sin^3 x}{3} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{(n-1) \cos^{n-2} x \cdot \sin^4 x}{3} \, dx \quad 1$$

$$= [0 - 0] + \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} \sin^2 x \left[\cos^{n-2} x \cdot \sin^2 x\right] dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\left[\cos^{n-2} x \sin^2 x\right] dx$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x - \cos^n x \sin^2 x \quad dx$$

1

$$3 I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$[3+(n-1)] I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n+2} I_{n-2} \quad \text{As required.}$$

1

(ii) $$I_2 = \frac{2-1}{2+2} I_0$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \quad dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\therefore 2\sin^2\theta = 1-\cos 2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

1

Must show.

$$= \frac{1}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right]$$

$$= \frac{\pi}{16} \quad \text{As required.}$$

(c) $z^5 = (z+1)^5$

(i) This is a polynomial of degree 4 (As z^5 cancels from both sides)
$\therefore$ It will have 4 roots over the complex field (Fundamental Theorem of Algebra)

(ii) $z^5 = (z+1)^5$

$$\frac{z^5}{(z+1)^5} = 1$$

let $\omega = \dfrac{z}{z+1}$ $\therefore$

$\therefore \quad \omega^5 = 1$

$\therefore \quad \omega = \text{cis}\left(\dfrac{2k\pi}{5}\right)$ where $k \in 0, \pm1, \pm2$

As $\omega = \dfrac{z}{z+1}$

let $\cos\dfrac{2k\pi}{5} = c$

$\sin\dfrac{2k\pi}{5} = s$

$\therefore \quad \omega = c + is$

$\omega z + \omega = z$

$\omega = z - \omega z$

$\omega = z(1-\omega)$

$z = \dfrac{\omega}{1-\omega}$

$$\therefore \quad z = \frac{c+is}{1+c+is} \times \frac{1+c-is}{1+c-is}$$

$$= \frac{c(1+c) - isc + i(1+c)s + s^2}{(1+c)^2 + s^2}$$

$$= \frac{c + c^2 + s^2 + i\left[s + sc - sc\right]}{(1+c)^2 + s^2}$$

$s^2 + c^2 = 1$

$$= \frac{(1+c) + is}{(1+c)^2 + s^2}$$

As $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$$z = \frac{1 + \cos^2\left(\frac{k\pi}{5}\right) - \sin^2\left(\frac{k\pi}{5}\right) + i\left(2\sin\left(\frac{k\pi}{5}\right)\cos\left(\frac{k\pi}{5}\right)\right)}{\left(1 + \cos^2\left(\frac{k\pi}{5}\right) - \sin^2\left(\frac{k\pi}{5}\right)\right)^2 + 4\sin^2\left(\frac{k\pi}{5}\right)\cos^2\left(\frac{k\pi}{5}\right)}$$

$$= \frac{2\cos^2\frac{k\pi}{5} + 2i\sin\left(\frac{k\pi}{5}\right)\cos\left(\frac{k\pi}{5}\right)}{\left(2\cos^2\frac{k\pi}{5}\right)^2 + 4\sin^2\left(\frac{k\pi}{5}\right)\cos^2\left(\frac{k\pi}{5}\right)}$$

$\div \cos^2\frac{k\pi}{5}$

$\div \cos^2\frac{k\pi}{5}$

$$= \frac{2 + 2i\dfrac{\left(\sin\frac{k\pi}{5}\right)}{\cos\left(\frac{k\pi}{5}\right)}}{4\cos^2\frac{k\pi}{5} + 4\sin^2\frac{k\pi}{5}}$$

$$= \frac{2 + 2i\cot\left(\frac{k\pi}{5}\right)}{4}$$

$$= \frac{1 + i\cot\left(\frac{k\pi}{5}\right)}{2}$$

(iii) If $z^5 = (z+1)^5$ has roots a_1, a_2, a_3 & a_4

$iz^5 = (iz+1)^5$

This is the same as $(iz)^5 = (iz+1)^5$

let $iz = \omega$

$\omega^5 = (\omega+1)^5$

$\downarrow$ This has solution $\omega = a_1, a_2, a_3, a_4$

$\therefore iz = a_1, a_2, a_3, a_4$

$\therefore z = \frac{a_1}{i}, \frac{a_2}{i}, \frac{a_3}{i}, \frac{a_4}{i}$

$= -ia_1, -ia_2, -ia_3, -ia_4$

$\therefore$ Solutions will be rotated $\frac{\pi}{2}$ clockwise. 1

1 or equivalent justification.